substitution of the previous expressions yields

$$\bar{X}_{cp} = \tfrac{2}{3}L \sec^2\theta$$

This result conforms more to the physical situation than does $X_{cp} = \frac{2}{3}L$ (which has been used without constraints other than $\alpha \leq \theta$).

Although $\bar{X}_{cp} = \frac{2}{3}L$ is a good approximation for small half-cone angles, large discrepancies will be evident if used for larger angles. Using this derived center of pressure location, the expression for the moment coefficient about the vertex now becomes

$$C_m = -\tfrac{1}{3}\sin 2\alpha \cot\theta$$

Figure 2 compares C_m obtained by using the above equation to that in Ref. 2. It may be seen that in the three cases shown, the moment coefficient from Ref. 2 is lower by 17%, 25%, and 41% for θ's of 20°, 30°, and 40°, respectively.

References

[1] Truitt, R. W., *Hypersonic Aerodynamics* (Ronald Press Company, New York, 1959), pp. 71–84.

[2] Dean, C. F., "Some comments on Newtonian impact theory," Air Force Systems Command, Kirtland Air Force Base, N. Mex. Air Force Weapons Lab. TR-64-176, 25–35 (June 1965).

Comments on "New Methods in Heat Flow Analysis"

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In a recent paper,[1] Biot applied variational principles to transient heat conduction equations and introduced the concepts of penetration depth and transit time. He derived expressions for these quantities using his proposed simplified procedures.

More recently, Goodman[2] also has assumed the concept of penetration depth as did McAdams.[3] To quote the latter, "When heating a relatively thick body for a relatively short time, it is clear that the heat will penetrate only a short distance in a zone near the surface and that the temperature at points farther below the surface would not be affected." This, unfortunately, is not physically correct and the author believes that the concepts of penetration depth and transit time should be treated with caution for this reason. Let us consider Biot's original example.

A slab of thickness l, with constant values of the thermal conductivity k and heat capacity C, has one face at $y = 0$ that is heated suddenly at time $t = 0$ to the temperature θ_0. The other face at $y = l$ is thermally insulated so that no heat flows across it. Biot assumed that the heat content distribution h, during the first heating phase, could be approximated by

$$\left.\begin{aligned} h &= h_0[1 - y/q]^2 \qquad \text{for} \quad y < q \\ h &= 0 \qquad\qquad\qquad \text{for} \quad y > q \end{aligned}\right\} \tag{1}$$

with $h = C\theta$, $h_0 = C\theta_0$, and the parameter q as the generalised coordinate which he called "the penetration depth." The transit time is the time for q to become equal to l, when the temperature at l first begins to rise.

Goodman[2] has not considered the previous example but his "heat balance integral technique" can be modified to give a solution. Goodman assumed as the temperature profile

$$\left.\begin{aligned} \theta &= \theta_0[1 - y/q]^3 \qquad \text{for} \quad y < q \\ \theta &= 0 \qquad\qquad\qquad \text{for} \quad y > q \end{aligned}\right\} \tag{2}$$

which was shown to satisfy all the boundary conditions at $y = 0$, and $y = q$.

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It is obvious, however, that the temperature profile

$$\left.\begin{aligned} \theta &= \theta_0[1 - y/q]^n \qquad \text{for} \quad y < q \\ \theta &= 0 \qquad\qquad\qquad \text{for} \quad y > q \end{aligned}\right\} \tag{3}$$

also satisfies the same boundary conditions, and it is instructive to examine the various results obtained using the methods of Biot,[1] Goodman,[2] and Kantorovich.[4] These are as follows:

Biot

$$q^2 = [(n + 1)(2n + 3)(3n + 1)/(5n + 3)]Kt \tag{4}$$

Goodman

$$q^2 = 2n(n + 1)Kt \tag{5}$$

Kantorovich

$$q^2 = n(2n + 1)Kt \tag{6}$$

where K = thermal diffusivity.

Table 1 presents values of the coefficients of Kt for the applicable range of n and it is seen that the agreement between the various results is good. Also, the penetration distance q obviously increases as n increases; and since during the first heating phase the temperature profile cannot remain constant, it is suggested that n is not constant but a function of time. If initially n is infinite the temperature distribution is exactly correct and the transit time is zero; this agrees with the exact analysis of the given problem[5] which yields

$$\theta = \theta_0[1 - f(y^2/4Kt)^{1/2}] \tag{7}$$

The term f is a function called Gauss' error integral which only equals unity at values of $y/2(Kt)^{1/2}$ equal to infinity, i.e., at $y = \infty$ or at $t = 0$. In other words, at finite values of y, the temperature rise for $t > 0$ is never zero. Thus, it is felt that the aforementioned concepts of transit time and penetration depth may be improved upon since the temperature profiles used as shown are not exact. Also, it is shown that the exponent n should be a function of time, and it is tentatively suggested that the following temperature profile be used for the first phase of heating

$$\theta = \theta_0[1 - y/l]^n \qquad \text{for} \qquad 0 \leq y \leq l \tag{8}$$

when $n = n_{(t)}$. At time $t = 0$, $n = \infty$, and eventually at the end of the first heating phase, the temperature profile can be approximated as in Ref. 1 by a parabolic profile, i.e., $n = 2$. Thus, a transit time can be defined as the time taken for n to decrease from ∞ to 2, at which point the second heating phase begins.

If Eq. (8) is used in the heat balance integral technique,[2] the resulting differential equation in n is

$$\dot{n} = -(K/l^2)\, n(n + 1)^2 \tag{9}$$

which yields the solution

$$Kt_1/l^2 = \log_e[(n + 1)/n] - 1/(n + 1) \tag{10}$$

This may be rewritten as

$$l^2 = F_1 K t_1 \tag{11}$$

Table 1 Values of coefficient F in the equation $q^2 = FKt$ for temperature distributions, $\theta = \theta_0[1 - y/q]^n$

n	A Biot[1]	B Goodman[2]	C Kantorovich[4]	D from Eq. (12)
1	5	4	...	...
2	11.3	12	10	13.8
3	20	24	21	26.5
4	31.1	40	36	43.2
5	41.6	60	55	63.9
∞	∞	∞	∞	∞

where t_1 is the transit time and the coefficient F_1 is

$$F_1 = \frac{1}{\log_e[(n+1)/n] - 1/(n+1)} \tag{12}$$

Values of F_1 are given in Table 1 for comparison, and it is seen that for $n = 2$ or 3 the agreement is reasonable. It should be noted that the values for columns B and D are both based on the heat balance integral technique and give, similarly, higher values than either the Biot or Kantorovich methods. Since Eq. (8) satisfies the physical situation better than Eq. (1)—Ref. 1, Eq. (2)—Ref. 2, or Eq. (3), and in view of the satisfactory results obtained as shown, it is suggested that Eq. (8) is preferable for such approximate methods of heat flow analysis and that the definition of transit time given after Eq. (8) is more realistic than that given in Ref. 1.

Incidentally, in Ref. 6, Chu has applied Biot's variational method to the problem of convective heating of a slab; and he shows that for small times after the onset of heating, a cubic temperature profile is better than a parabolic profile—although at large times the reverse is true. This result suggests also that the exponent n should be considered a function of time.

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[3] McAdams, W., *Heat Transmission* (McGraw-Hill Book Company Inc., New York, 1954), 3rd ed., p. 39.

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[5] Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids* (Clarendon Press, Oxford, England, 1948), 2nd ed., p. 100.

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